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***Probability and Mathematical Statistics* 2016, 36(1), 21-33.**

PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 36, Fasc. 1 (2016), pp. 21–33

ON THE MOMENT DETERMINACY OF PRODUCTS OF NON-IDENTICALLY DISTRIBUTED RANDOM VARIABLES

BY

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Abstract. We show first that there are intrinsic relationships among different conditions, old and recent, which lead to some general statements in both the Stieltjes and the Hamburger moment problems. Then we describe checkable conditions and prove new results about the moment (in)determinacy for products of independent and non-identically distributed random variables. We treat all three cases: when the random variables are nonnegative (Stieltjes case), when they take values in the whole real line (Hamburger case), and the mixed case. As an illustration we characterize the moment determinacy of products of random variables whose distributions are generalized gamma or double generalized gamma all with distinct shape parameters. Among other corollaries, the product of two independent random variables, one exponential and one inverse Gaussian, is moment determinate, while the product is moment indeterminate for the cases: one exponential and one normal, one chi-square and one normal, and one inverse Gaussian and one normal.

2010 AMS Mathematics Subject Classification: Primary: 60E05, 44A60; Secondary: 62E10.

Key words and phrases: Product of random variables, Stieltjes moment problem, Hamburger moment problem, Cramér's condition, Carleman's condition, Krein's condition, Hardy's condition.

1. INTRODUCTION

There is a long standing interest in studying products of random variables; see, e.g., [13], [16], [23], [6], [22], [10], [27] and the references therein. The reasons are twofold. On one hand, to deal with products leads to non-trivial, difficult and challenging theoretical problems requiring to use diverse ideas and techniques. Let us mention just a few sources: [10], [1], [3]. On the other hand, products of random variables are naturally involved in stochastic modelling of complex random phenomena in areas such as statistical physics, quantum theory, communication theory and financial modelling; see, e.g., [4], [7]–[10], [12], [21], [5].

In general, it is rare to find explicit closed-form expressions for the densities or the distributions of products of random variables with different distributions. It is, however, possible to study successfully the moment problem for products of independent random variables; see, e.g., [15], [26]. Answers about the moment (in)determinacy can be found if requiring only information about the asymptotics of the moments or about the tails of the densities or of their distributions.

All random variables considered in this paper are defined on an underlying probability space $(\Omega, \mathcal{F}, \mathbf{P})$, and we denote by $\mathbf{E}[X]$ the expected value of the random variable X. A basic assumption is that the random variables we deal with have finite moments of all positive orders, i.e. $\mathbf{E}[|X|^k] < \infty,\ k = 1, 2, \ldots$ We write $X \sim F$ to mean that X is a random variable whose distribution function is F and denote its kth order moment by $m_k = \mathbf{E}[X^k]$. We say that X or F is *moment determinate* (M-det) if F is the only distribution having the moment sequence $\{m_k\}_{k=1}^{\infty}$; otherwise, we say that X or F is *moment indeterminate* (M-indet). We use traditional notions, notation and terms such as Cramér's condition, Carleman's condition, Krein's condition, and Hardy's condition (see, e.g., [17], [15], and [26]).

We use $\Gamma(\cdot)$ for the Euler-gamma function, $\mathbb{R} = (-\infty, \infty)$ for the set of all real numbers, $\mathbb{R}_+ = [0, \infty)$ for the nonnegative numbers, the symbol $\mathcal{O}(\cdot)$ with its usual meaning in asymptotic analysis, and the abbreviation i.i.d. for independent and identically distributed (random variables).

In Section 2 we describe useful intrinsic relationships among different old and recent conditions involved in the Stieltjes and/or the Hamburger moment problems. Then we present some new results under conditions which are relatively easy to check. In Section 3 we deal with the moment determinacy of products of independent nonnegative random variables with different distributions, while in Section 4 we consider products of random variables with values in $\mathbb{R}$. Finally, in Section 5, we treat the mixed case: products of both types of random variables, nonnegative ones and real ones, the latter with values in $\mathbb{R}$.

The results presented in this paper extend some previous results for products of i.i.d. random variables. Here we need a more refined analysis of the densities of products than in the i.i.d. case. As an illustration we characterize the moment (in)determinacy of products of random variables whose distributions are generalized gamma or double generalized gamma all with distinct shape parameters. We have derived several corollaries involving popular distributions widely used in theoretical studies and applications. Let us list a few:

(i) the product of two independent random variables, one exponential and one inverse Gaussian, is M-det;

(ii) the product of independent exponential and normal random variables is M-indet;

(iii) the product of independent chi-square and normal random variables is M-indet; and

(iv) the product of independent inverse Gaussian and normal random variables is M-indet.

2. SOME GENERAL RESULTS

In this section we present two lemmas, each containing workable conditions which, more or less, are available in the literature. Some of these conditions are old, while others are recent. We describe intrinsic relationships among these conditions and use them to obtain new results; see Theorems 2.1–2.4.

Our findings in this section can be considered as a useful complement to the classical criteria of Cramér, Carleman, Krein, Hardy and their converses, so that all these taken together make more clear, and possibly complete, the picture of what is in our hands when discussing the determinacy of distributions in terms of their moments.

2.1. Stieltjes case. We present the first important lemma.

LEMMA 2.1. *Let* $0 \leqslant X \sim F$. *Then the following statements are equivalent*:

(i) $m_k = \mathcal{O}(k^{2k})$ *as* $k \to \infty$.

(ii) $\limsup_{k\to\infty} k^{-1}\, m_k^{1/(2k)} < \infty$.

(iii) $m_k \leqslant c_0^k\,(2k)!,\ k = 1, 2, \ldots,$ *for some constant* $c_0 > 0$.

(iv) X *satisfies Hardy's condition*, *namely*, $\mathbf{E}[e^{c\sqrt{X}}] < \infty$ *for some constant* $c > 0$.

The equivalence of conditions (i) and (ii), a known fact for decades, can be easily checked. Conditions (iii) and (iv) appeared recently and their equivalence to condition (ii) was shown in [25].

THEOREM 2.1. *Let* $0 \leqslant X \sim F$ *with moments growing as follows*: $m_k = \mathcal{O}(k^{ak})$ *as* $k \to \infty$ *for some constant* $a \in (0, 2]$. *Then the following statements hold*:

(i) X *satisfies Hardy's condition*, *and hence* X *is M-det.*

(ii) *The boundary value* $a = 2$ *is the best possible for* X *to be M-det. In fact*, *there is an M-indet random variable* $\tilde{X} \geqslant 0$ *such that* $\mathbf{E}[\tilde{X}^k] = \mathcal{O}(k^{ak})$ *as* $k \to \infty$ *for all* $a > 2$.

P r o o f. Part (i) follows easily from Lemma 2.1. To prove the first conclusion of part (ii), we may consider the positive random variable X_a with density $f_a(x) = \big(\Gamma(a+1)\big)^{-1}\exp(-x^{1/a}),\ x > 0$, where $a > 0$. Then the conclusion follows from the facts: (i) X_a is M-indet if $a > 2$ (see Theorem 2 in [19]) and (ii) for $a \in (0, e)$, $\mathbf{E}[X_a^k] = \Gamma\big((k+1)a\big)/\Gamma(a) = \mathcal{O}(k^{ak})$ as $k \to \infty$. As for the stronger second conclusion of part (ii), we consider the *universal* $\tilde{X}$ (independent of a) with density $\tilde{f}(x) = \tilde{c}\,\exp\big(-\sqrt{x}/(1 + |\ln x|^{\delta})\big),\ x > 0$, where $\delta > 1$ and $\tilde{c}$ is the norming constant. Then it can be shown that $\tilde{X}$ is M-indet and, for all $a > 2$, $\mathbf{E}[\tilde{X}^k] = \mathcal{O}(k^{ak})$ as $k \to \infty$. This completes the proof. ∎

REMARK 2.1. *For* $0 \leqslant X \sim F$, *let us compare the following two moment conditions*: (a) $m_k = \mathcal{O}(k^{2k})$ *as* $k \to \infty$, *and* (b) $m_{k+1}/m_k = \mathcal{O}\big((k+1)^2\big)$ *as* $k \to \infty$. *Here* (a) *is the condition in Theorem* 2.1, *while condition* (b) *was introduced and used in the recent paper* [15]. *Both conditions are checkable and each of them guarantees the moment determinacy of F. Just to mention that condition* (b) *implies condition* (a) *by referring to Theorem* 3 *in* [15], *while the converse may not be true in general.*

The next result, Theorem 2.2 below, is the converse of Theorem 2.1, and deals with the moment indeterminacy of nonnegative random variables. First we need one condition which is used a few times in the sequel.

CONDITION L. Suppose, in the Stieltjes case, that $f(x),\ x \in \mathbb{R}_+$, is a density function such that, for some fixed $x_0 > 0$, f is strictly positive and differentiable for $x > x_0$ and

$$L_f(x) := -\frac{xf'(x)}{f(x)} \nearrow \infty \quad \text{as } x_0 < x \to \infty.$$

In the Hamburger case we require the density $f(x),\ x \in \mathbb{R}$, to be symmetric.

This condition plays a significant rôle in moment problems for absolutely continuous probability distributions. It was explicitly introduced and efficiently used for the first time in [14] and later used by several authors naming it as 'Lin's condition'. This condition is involved in some of our results to follow.

THEOREM 2.2. *Let* $0 \leqslant X \sim F$ *and its moment sequence* $\{m_k,\ k = 1, 2, \dots\}$ *grow 'fast' in the sense that* $m_k \geqslant c\,k^{(2+\varepsilon)k},\ k = 1, 2, \dots,$ *for some constants* $c > 0$ *and* $\varepsilon > 0$. *Assume further that* X *has a density function* f *which satisfies the above Condition* L. *Then* X *is M-indet.*

Proof. By the condition on the moments, the Carleman quantity for the moments of F is finite. Then, applying Condition L and the second part of the proof of Theorem 4 in [15], we conclude that indeed X is M-indet. ∎

REMARK 2.2. *To provide one application of Theorem* 2.2, *let us consider, for example, the random variable* $X = \xi^{2(1+\varepsilon)}$, *where* $\varepsilon > 0$ *and* $\xi \sim Exp(1)$, *the standard exponential distribution. On one hand, we can use the Krein criterion and show that X is M-indet. On the other hand, X satisfies the moment condition in Theorem* 2.2. *And here is the point: instead of applying Krein's condition, we can prove the moment indeterminacy of X by checking that its density f satisfies Condition* L. *In general, we follow the approach which is easier.*

2.2. Hamburger case. We start with Lemma 2.2 establishing the equivalence of different type of conditions involved to decide whether a distribution on the whole real line $\mathbb{R}$ is M-det. Then we present some new results. Theorem 2.3 below is a slight modification, in a new light, of a result in [2], p. 92, while Theorem 2.4 is the converse of Theorem 2.3. Both proofs are omitted.

LEMMA 2.2. *Let X be a random variable taking values in $\mathbb{R}$. Then the following statements are equivalent*:

(i) $m_{2k} = \mathcal{O}\big((2k)^{2k}\big)$ *as* $k \to \infty$.

(ii) $\limsup_{k\to\infty}(2k)^{-1}\, m_{2k}^{1/(2k)} < \infty$.

(iii) $m_{2k} \leqslant c_0^k\,(2k)!,\ k = 1, 2, \ldots,$ *for some constant* $c_0 > 0$.

(iv) *X satisfies Cramér's condition*: *its moment generating function exists.*

Proof. It is easy to check the equivalence of conditions (i) and (ii). The equivalence of conditions (ii) and (iv) is well known, but we provide here a simple and instructive proof based on condition (i). Indeed, by Lemma 2.1 above, condition (i) is equivalent to say that the random variable $Y = X^2$ satisfies Hardy's condition, namely, $\mathbf{E}[\exp(c\sqrt{Y})] = \mathbf{E}[\exp(c|X|)] < \infty$ for some constant $c > 0$. The latter, however, means that X itself has a moment generating function. This is exactly the statement (iv). Finally, applying again Lemma 2.1 to the nonnegative random variable Y, we obtain the equivalence of (ii) and (iii). Therefore, as stated, all four conditions (i)–(iv) are equivalent. ■

THEOREM 2.3. *Let $X \sim F$, where F has an unbounded support* $\operatorname{supp}(F) \subset \mathbb{R}$ *and its moments satisfy the condition*: $m_{2k} = \mathcal{O}\big((2k)^{2ak}\big)$ *as* $k \to \infty$ *for some constant* $a \in (0, 1]$. *Then the following statements hold*:

(i) *X satisfies Cramér's condition*, *and hence is M-det.*

(ii) *The boundary value $a = 1$ is the best possible for X to be M-det. In fact*, *there is an M-indet random variable $\tilde{X}$ such that* $\mathbf{E}[\tilde{X}^{2k}] = \mathcal{O}\big((2k)^{2ak}\big)$ *as* $k \to \infty$ *for all* $a > 1$.

REMARK 2.3. *Let $X \sim F$ with F having unbounded support*, $\operatorname{supp}(F) \subset \mathbb{R}$. *We want to compare the following two moment conditions*: (a) $m_{2k} = \mathcal{O}\big((2k)^{2k}\big)$ *as* $k \to \infty$, *and* (b) $m_{2(k+1)}/m_{2k} = \mathcal{O}\big((k+1)^2\big)$ *as* $k \to \infty$. *Here* (a) *is the condition of the growth of the moments stated in Theorem* 2.3, *while condition* (b) *was introduced and successfully exploited in the recent work* [26]. *Both conditions are checkable and each of them guarantees the moment determinacy of X and F. Let us mention that condition* (b) *implies condition* (a) *by referring to Theorem* 2 *in* [26], *while the converse may not in general be true.*

THEOREM 2.4. *Suppose that the moments of $X \sim F$ grow 'fast' in the sense that* $m_{2k} \geqslant c(2k)^{2(1+\varepsilon)k},\ k = 1, 2, \ldots,$ *for some positive constants c and ε. Assume further that X has a density function f which is symmetric about zero and satisfies the above Condition* L. *Then X is M-indet.*

REMARK 2.4. *For example*, *instead of applying Krein's condition*, *we can use Theorem* 2.4 *to prove the moment indeterminacy of $X \sim F$ whose density is the symmetrization of that of* $\xi^{1+\varepsilon}$, *where* $\varepsilon > 0$ *and* $\xi \sim Exp(1)$.

3. PRODUCTS OF NONNEGATIVE RANDOM VARIABLES

We start with two results describing relatively simple conditions on the random variables $\xi_1, \ldots, \xi_n$ in order to guarantee that their product is M-det.

THEOREM 3.1. *Suppose that the moments* $m_{i,k} = \mathbf{E}[\xi_i^k],\ i = 1, \ldots, n$, *of the independent nonnegative random variables* $\xi_1, \ldots, \xi_n$ *satisfy the conditions*:

$$m_{i,k} = \mathcal{O}(k^{a_i k}) \quad \textit{as}\ \ k \to \infty,\ \ \textit{for}\ \ i = 1, \ldots, n,$$

where $a_1, \ldots, a_n$ *are positive constants. If the constants* $a_1, \ldots, a_n$ *are such that* $a_1 + \ldots + a_n \leqslant 2$, *then the product* $Z_n = \xi_1 \ldots \xi_n$ *is M-det.*

Proof. With $m_k = \mathbf{E}[Z_n^k]$ we infer, by the independence of ξ_i, that

$$m_k = m_{1,k} \ldots m_{n,k} = \mathcal{O}(k^{a_1 k}) \ldots \mathcal{O}(k^{a_n k}) = \mathcal{O}(k^{ak}) \quad \text{as}\ \ k \to \infty,$$

where $a = a_1 + \ldots + a_n$. Since, by assumption, $a \leqslant 2$, we apply Theorem 2.1 (i) to conclude the M-det property of the product Z_n. ■

Similarly, we have the following result in terms of the ratio of moments.

THEOREM 3.2. *Suppose that the growth rates* $r_1, \ldots, r_n$ *of the moments of the independent nonnegative random variables* $\xi_1, \ldots, \xi_n$ *satisfy*

$$\frac{m_{1,k+1}}{m_{1,k}} = \mathcal{O}\big((k+1)^{r_1}\big),\ \ldots,\ \frac{m_{n,k+1}}{m_{n,k}} = \mathcal{O}\big((k+1)^{r_n}\big) \quad \textit{as}\ \ k \to \infty,$$

where $m_{i,k} = \mathbf{E}[\xi_i^k],\ i = 1, \ldots, n,\ k = 1, 2, \ldots$ *If the rates* $r_1, \ldots, r_n$ *are such that* $r_1 + \ldots + r_n \leqslant 2$, *then the product* $Z_n = \xi_1 \ldots \xi_n$ *is M-det.*

Let us provide now conditions under which the product Z_n becomes M-indet.

THEOREM 3.3. *Let us consider* n *independent nonnegative random variables* $\xi_i \sim F_i,\ i = 1, \ldots, n$, *where* $n \geqslant 2$. *Suppose that each* F_i *has a density* $f_i > 0$ *on* $(0, \infty)$ *and that the following conditions are satisfied*:

(i) *At least one of the densities* $f_1(x), \ldots, f_n(x)$ *is decreasing in* $x \geqslant x_0$, *where* $x_0 \geqslant 1$ *is a constant.*

(ii) *For each* $i = 1, \ldots, n$, *there exists a constant* $A_i > 0$ *such that the density* f_i *and the tail function* $\overline{F_i} = 1 - F_i$ *satisfy the relation*

$$f_i(x)/\overline{F_i}(x) \geqslant A_i/x \quad \textit{for}\ \ x \geqslant x_0, \tag{3.1}$$

and there exist constants $B_i > 0,\ \alpha_i > 0,\ \beta_i > 0$ *and real* γ_i *such that*

$$\overline{F_i}(x) \geqslant B_i x^{\gamma_i} \exp(-\alpha_i x^{\beta_i}) \quad \textit{for}\ \ x \geqslant x_0. \tag{3.2}$$

If, in addition to conditions (i) *and* (ii), *the parameters* $\beta_1, \ldots, \beta_n$ *are such that* $\sum_{i=1}^n 1/\beta_i > 2$, *then the product* $Z_n = \xi_1 \ldots \xi_n$ *is M-indet.*

Proof. We may assume, by condition (i), that f_n is the density which is decreasing in $x \geqslant x_0$. Then, clearly, Z_n is nonnegative and its density, say h_n, can be written as follows: for $x > 0$,

$$h_n(x) = \int_0^\infty \int_0^\infty \dots \int_0^\infty \frac{f_1(u_1)}{u_1} \frac{f_2(u_2)}{u_2} \dots \frac{f_{n-1}(u_{n-1})}{u_{n-1}} \times f_n\left(\frac{x}{u_1 u_2 \dots u_{n-1}}\right) du_1 du_2 \dots du_{n-1}.$$

This representation shows that $h_n(x) > 0$. To prove the M-indet property of Z_n, we will show that the Krein quantity $\mathrm{K}[h_n]$ is finite. Thus, we need an estimate of the lower bound of h_n.

To do this, let us define $B = \sum_{i=1}^n \beta_i^{-1},\ \theta_i = \beta_i^{-1}/B \in (0,1),\ i = 1, \dots, n$, $\theta = \min\{\theta_1, \dots, \theta_n\}$ and $x_\theta = (2^{n-1}x_0)^{1/\theta}$. Then, for each $x > x_\theta$, we take $a_i = x^{\theta_i} \geqslant x_0,\ i = 1, \dots, n-1$, which together imply that

$$x/(2^{n-1}a_1 a_2 \dots a_{n-1}) = x^{\theta_n}/2^{n-1} \geqslant x_0,$$

because $\sum_{i=1}^n \theta_i = 1$. For these x and a_i, we have, by condition (i), the following:

$$\begin{aligned} h_n(x) &\geqslant \int_{a_1}^{2a_1} \int_{a_2}^{2a_2} \dots \int_{a_{n-1}}^{2a_{n-1}} \frac{f_1(u_1)}{u_1} \frac{f_2(u_2)}{u_2} \dots \frac{f_{n-1}(u_{n-1})}{u_{n-1}} \times f_n\left(\frac{x}{u_1 u_2 \dots u_{n-1}}\right) du_1 du_2 \dots du_{n-1} \\ &\geqslant f_n\left(\frac{x}{a_1 a_2 \dots a_{n-1}}\right) \prod_{i=1}^{n-1} \int_{a_i}^{2a_i} \frac{f_i(u)}{u} du. \end{aligned}$$

Then, by Lemma 3.1 below (with $r = 2$) and (3.1) and (3.2), we have, for $x > x_\theta$,

$$h_n(x) \geqslant f_n(x^{\theta_n}) \prod_{i=1}^{n-1} \frac{A_i}{2(1+A_i)} \frac{\overline{F_i}(a_i)}{a_i} \geqslant C x^\gamma \exp\big[-\sum_{i=1}^n \alpha_i x^{\theta_i \beta_i}\big],$$

where $C = 2^{1-n} A_n \big(\prod_{i=1}^{n-1} A_i/(1+A_i)\big) \prod_{i=1}^n B_i$ and $\gamma = \sum_{i=1}^n \theta_i(\gamma_i - 1)$.

We now evaluate the Krein quantity $\mathrm{K}[h_n]$ on (x_θ, ∞). Recall that this is a Stieltjes case and we have the following:

$$\mathrm{K}[h_n] = \int_{x_\theta}^\infty \frac{-\log h_n(x^2)}{1+x^2} dx < \infty.$$

The conclusion about the finiteness of $\mathrm{K}[h_n]$ relies essentially on the facts that $\theta_i \beta_i = 1/B < 1/2,\ i = 1, 2, \dots, n$. Therefore, Z_n is M-indet by Proposition 1 in [17]. The proof is complete. ■

LEMMA 3.1. *Let F be a distribution on $\mathbb{R}$ such that* (i) *it has density f on the subset $[a, ra]$, where $a > 0$ and $r > 1$, and* (ii) *for some constant $A > 0$, $f(x)/\overline{F}(x) \geqslant A/x$ on $[a, ra]$. Then*

$$\int_a^{ra} \frac{f(x)}{x} dx \geqslant \left(1 - \frac{1}{r}\right) \frac{A}{1+A} \frac{\overline{F}(a)}{a}.$$

Proof. Integration by parts yields

$$\begin{aligned}\int_a^{ra} \frac{f(x)}{x} dx &= -\int_a^{ra} \frac{d\overline{F}(x)}{x} = \frac{\overline{F}(a)}{a} - \frac{\overline{F}(ra)}{ra} - \int_a^{ra} \frac{\overline{F}(x)}{x^2} dx \\ &\geqslant \left(1 - \frac{1}{r}\right) \frac{\overline{F}(a)}{a} - \frac{1}{A} \int_a^{ra} \frac{f(x)}{x} dx.\end{aligned}$$

The last inequality is due to the monotonicity of F and the condition on the failure rate $f/\overline{F}$. Hence the required conclusion follows. ∎

EXAMPLE 3.1. For illustration of how to use Theorem 3.3, consider the class of generalized gamma distributions. We use the notation $\xi \sim GG(\alpha, \beta, \gamma)$ if the density function of the random variable ξ is of the form

$$f(x) = cx^{\gamma-1} \exp(-\alpha x^{\beta}), \quad x \geqslant 0. \tag{3.3}$$

Here $\alpha, \beta, \gamma > 0$, $f(0) = 0$ if $\gamma \neq 1$, and $c = \beta\alpha^{\gamma/\beta}/\Gamma(\gamma/\beta)$ is the norming constant. We have the following statement (see also Theorem 8.4 in [18] for a more general result with different proof).

COROLLARY 3.1. *Suppose $\xi_1, \ldots, \xi_n$ are n independent random variables such that $\xi_i \sim GG(\alpha_i, \beta_i, \gamma_i),\ i = 1, \ldots, n$, and let $Z_n = \xi_1 \ldots \xi_n$. Then Z_n is M-det if and only if $\sum_{i=1}^n 1/\beta_i \leqslant 2$.*

Proof. Note that for $\xi \sim GG(\alpha, \beta, \gamma)$ defined by (3.3) we have two properties: (a) $f(x)/\overline{F}(x) \approx \alpha\beta x^{\beta-1}$, $\overline{F}(x) \approx [c/(\alpha\beta)] x^{\gamma-\beta} \exp(-\alpha x^{\beta})$ as $x \to \infty$, and (b) $m_k = \alpha^{-k/\beta}\Gamma\big((\gamma+k)/\beta\big)/\Gamma(\gamma/\beta) = \mathcal{O}(k^{k/\beta})$ as $k \to \infty$. Hence the sufficiency part follows from Theorem 2.1 because $\mathbf{E}[Z_n^k] = \mathcal{O}(k^{Bk})$ as $k \to \infty$, where $B = \sum_{i=1}^n 1/\beta_i$. The necessity part is a consequence of Theorem 3.3. ∎

EXAMPLE 3.2. Consider the class of inverse Gaussian distributions. We say that $X \sim IG(\mu, \lambda)$ if the density of X is of the form

$$f(x) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \exp\left[-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right], \quad x > 0, \tag{3.4}$$

where $\mu, \lambda > 0$ and $f(0) = 0$. If $X \sim IG(\mu, \lambda)$, then it has a moment generating function. This in turn implies that the power $Y = X^2$ satisfies Hardy's condition, and hence is M-det. Actually, it follows that, for real r, X^r is M-det if and only if $|r| \leqslant 2$ (see [24]). If ξ_1 and ξ_2 are two i.i.d. random variables with density (3.4), then the product $Z = \xi_1\xi_2$ is also M-det due to Proposition 1 (iii) in [15]. The next result is for products of non-identically distributed random variables.

COROLLARY 3.2. *Let $\xi_1 \sim IG(\mu_1, \lambda_1)$, $\xi_2 \sim IG(\mu_2, \lambda_2)$ and $\eta \sim Exp(1)$ be three independent random variables. Then the following statements hold*:

(i) *$Z = \xi_1\eta$ is M-det.*

(ii) *$Z = \xi_1\xi_2$ is M-det.*

(iii) *$Z = \xi_1\xi_2\eta$ is M-indet.*

Proof. First, for $X \sim F = IG(\mu, \lambda)$, it can be shown (we omit the details) that the moment $\mathbf{E}[X^k] = \mathcal{O}(k^k)$ as $k \to \infty$. Second, the hazard rate function $r(x) = f(x)/\overline{F}(x) \to \lambda/(2\mu^2) > 0$ as $x \to \infty$. Third, the tail function $\overline{F}$ satisfies (3.2) with the exponent $\beta = 1$. With these three steps we are in a position to apply Theorems 3.1 and 3.3 to confirm the validity of (i)–(iii) as stated above. ∎

4. PRODUCTS OF RANDOM VARIABLES IN $\mathbb{R}$

We start with two results describing relatively simple conditions on the random variables $\xi_1, \ldots, \xi_n$ in order to guarantee that their product is M-det. The results are similar to the above Theorems 3.1 and 3.2, however, we remember that here we deal with the Hamburger case, so we work with the even order moments.

THEOREM 4.1. *Suppose that the even order moments $m_{i,2k} = \mathbf{E}[\xi_i^{2k}], i = 1, \ldots, n,$ of the independent random variables $\xi_1, \ldots, \xi_n$ satisfy the conditions*:

$$m_{i,2k} = \mathcal{O}\big((2k)^{2a_i k}\big) \quad \textit{as } k \to \infty, \textit{ for } i = 1, \ldots, n,$$

where $a_1, \ldots, a_n$ are positive constants. If the constants $a_1, \ldots, a_n$ are such that $a_1 + \ldots + a_n \leqslant 1$, then the product $Z_n = \xi_1 \ldots \xi_n$ is M-det.

Proof. With $m_{2k} = \mathbf{E}[Z_n^{2k}]$ we have, by the independence of ξ_i,

$$m_{2k} = m_{1,2k} \ldots m_{n,2k} = \mathcal{O}\big((2k)^{2a_1 k}\big) \ldots \mathcal{O}\big((2k)^{2a_n k}\big) = \mathcal{O}\big((2k)^{2ak}\big) \text{ as } k \to \infty,$$

where $a = a_1 + \ldots + a_n$. Since, by assumption, $a \leqslant 1$, we apply Theorem 2.3 (i) to conclude the M-det property of the product Z_n. ∎

A similar result holds in terms of the ratio of even order moments.

THEOREM 4.2. *Suppose that the growth rates $r_1, \ldots, r_n$ of the even order moments of the independent random variables $\xi_1, \ldots, \xi_n$ satisfy*

$$\frac{m_{1,2(k+1)}}{m_{1,2k}} = \mathcal{O}\big((k+1)^{r_1}\big), \ \ldots, \ \frac{m_{n,2(k+1)}}{m_{n,2k}} = \mathcal{O}\big((k+1)^{r_n}\big) \quad \textit{as } k \to \infty,$$

where $m_{i,2k} = \mathbf{E}[\xi_i^{2k}], i = 1, \ldots, n,\ k = 1, 2, \ldots$ *If the rates* $r_1, \ldots, r_n$ *are such that* $r_1 + \ldots + r_n \leqslant 2$, *then the product* $Z_n = \xi_1 \ldots \xi_n$ *is M-det.*

Let us describe now conditions under which the product Z_n is M-indet.

THEOREM 4.3. *Let us consider* n *independent random variables* $\xi_i \sim F_i, i = 1, \ldots, n$, *where* $n \geqslant 2$. *Suppose each* F_i *has a density* f_i *which is strictly positive on* $\mathbb{R}$ *and symmetric about zero. Assume further that the following conditions are satisfied*:

(i) *At least one of the densities* $f_1(x), \ldots, f_n(x)$ *is decreasing in* $x \geqslant x_0$, *where* $x_0 \geqslant 1$ *is a constant.*

(ii) *For each* $i = 1, \ldots, n$, *there exists a constant* $A_i > 0$ *such that* (3.1) *holds and there exist constants* $B_i > 0,\ \alpha_i > 0, \beta_i > 0$ *and real* γ_i *such that* (3.2) *holds.*

If, in addition to the above conditions, $\sum_{i=1}^{n} 1/\beta_i > 1$, *then the product* $Z_n = \xi_1 \ldots \xi_n$ *is M-indet.*

Proof. We may assume, by condition (i), that f_n is the density which is decreasing in $x \geqslant x_0$. Then the density h_n of Z_n is symmetric about zero (see, e.g., [11]) and h_n can be written as follows: for $x > 0$,

$$(4.1)\quad h_n(x) = 2^{n-1} \int_0^\infty \int_0^\infty \ldots \int_0^\infty \frac{f_1(u_1)}{u_1} \frac{f_2(u_2)}{u_2} \ldots \frac{f_{n-1}(u_{n-1})}{u_{n-1}} \times f_n\left(\frac{x}{u_1 u_2 \ldots u_{n-1}}\right) du_1 du_2 \ldots du_{n-1}.$$

Hence $h_n(x) > 0$. The remaining proof is similar to that of Theorem 3.3 (by using Theorem 2.2 in [20] for the Hamburger case) and is omitted. ∎

EXAMPLE 4.1. We now apply Theorem 4.3 to the product of double generalized gamma random variables. We write $\xi \sim DGG(\alpha, \beta, \gamma)$ if ξ is a random variable in $\mathbb{R}$ with density function of the form

$$(4.2)\qquad f(x) = c|x|^{\gamma-1} \exp(-\alpha|x|^{\beta}), \quad x \in \mathbb{R}.$$

Here $\alpha, \beta, \gamma > 0,\ f(0) = 0$ if $\gamma \neq 1$, and $c = \beta\alpha^{\gamma/\beta}/\big(2\Gamma(\gamma/\beta)\big)$ is a norming constant.

COROLLARY 4.1. *Suppose* $\xi_1, \ldots, \xi_n$ *are* n *independent random variables, and let* $\xi_i \sim DGG(\alpha_i, \beta_i, \gamma_i),\ i = 1, \ldots, n$. *Then the product* $Z_n = \xi_1 \ldots \xi_n$ *is M-det if and only if* $\sum_{i=1}^{n} 1/\beta_i \leqslant 1$.

Proof. Note that for the moment $m_{2k} = \mathbf{E}[\xi^{2k}]$ of $\xi \sim DGG(\alpha, \beta, \gamma)$ defined by (4.2) we have the following relation: $m_{2k} = \mathcal{O}\big((2k)^{2k/\beta}\big)$ as $k \to \infty$. Thus, the sufficiency part is exactly Theorem 10 in [26]. The same statement can also be proved by Theorem 4.1 above. Finally, the necessity part follows from Theorem 4.3 (we may redefine $f(0)$ to be a positive number if necessary). ∎

5. THE MIXED CASE

For completeness of our study we need to consider products of both types of random variables, nonnegative ones and real ones. Since such a 'mixed' product takes values in $\mathbb{R}$, this is a Hamburger case, so we can formulate results similar to Theorems 4.1 and 4.2. Since the conditions, the statements and the arguments are almost as in these two theorems, we do not give details. Instead, we present now a result in which the 'mixed' product $Z_n = \xi_1 \dots \xi_n$ is M-indet.

THEOREM 5.1. *Given are n independent random variables such that the 'first' group, $\xi_1, \dots, \xi_{n_0}$, consists of nonnegative variables, while the variables in the 'second' group, $\xi_{n_0+1}, \dots, \xi_n$, all take values in $\mathbb{R}$, where $1 \leqslant n_0 < n$. Suppose each $\xi_i \sim F_i$ has a density f_i and assume further that $f_i,\ i = 1, \dots, n_0$, are strictly positive on $(0, \infty)$, while $f_j,\ j = n_0 + 1, \dots, n$, are strictly positive on $\mathbb{R}$ and symmetric about zero. Moreover, assume the following conditions are satisfied*:

(i) *At least one of the densities $f_j(x),\ j = n_0 + 1, \dots, n$, is decreasing in $x \geqslant x_0$, where $x_0 \geqslant 1$ is a constant.*

(ii) *For each $i = 1, \dots, n$, there exists a constant $A_i > 0$ such that* (3.1) *holds and there exist constants $B_i > 0,\ \alpha_i > 0,\ \beta_i > 0$ and real γ_i such that* (3.2) *holds.*

If, *in addition to* (i) *and* (ii), *the parameters β_i are such that $\sum_{i=1}^{n} 1/\beta_i > 1$, then the product $Z_n = \xi_1 \dots \xi_n$ is M-indet.*

To prove the theorem, it suffices to replace the coefficient 2^{n-1} in the integral form of h_n in (4.1) by 2^{n-n_0-1}. As an application of Theorem 5.1 we derive below two interesting corollaries.

COROLLARY 5.1. *Consider two independent random variables*, ξ *and* η, *where $\xi \sim Exp(1)$ and $\eta \sim \mathcal{N}(0, 1)$* (*standard normal*). *Then $Z = \xi\, \eta$ is M-indet.*

COROLLARY 5.2. (i) *The product of two independent random variables*, *one chi-square and one normal*, *is M-indet.*

(ii) *The product of two independent random variables*, *one inverse Gaussian and one normal*, *is M-indet.*

Acknowledgments. The authors would like to thank the Editor, Professor Władysław Szczotka, and the anonymous Referee for helpful comments and suggestions on a previous longer version of the paper. In particular, the Referee suggested using the density f_a in the neat proof of Theorem 2.1 and adding the references [18] and [19].

The work of GDL was partly supported by the Ministry of Science and Technology of Taiwan (ROC) under Grant NSC 102-2118-M-001-008-MY2.

JS was supported by the International Laboratory of Quantitative Finance (Higher School of Economics, Moscow) under Grant No. 14.12.31.0007 to visit Moscow, March 2014, when a part of this work was completed. The work was finalized during the visit of JS to Academia Sinica (Taipei, Taiwan, ROC), May 2014, with a support from the Emeritus Fellowship provided by Leverhulme Trust (UK).

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Received on 30.11.2014;
revised version on 24.2.2015